

The Collective Field theory of a Singular Supersymmetric Matrix Model

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The supersymmetric collective field theory with the potential $v'(x) = \omega x - \frac{\eta}{x}$ is studied, motivated by the matrix model proposed by Jevicki and Yoneya to describe two dimensional string theory in a black hole background. Consistency with supersymmetry enforces a two band solution. A supersymmetric classical configuration is found, and interpreted in terms of the density of zeros of certain Laguerre polynomials. The spectrum of the model is then studied and is seen to correspond to a massless scalar and a majorana fermion. The x space eigenfunctions are constructed and expressed in terms of Chebyshev polynomials. Higher order interactions are also discussed.

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I. INTRODUCTION

The collective field theory [1] description of the quantum mechanics of single matrix systems has had important applications in QCD type models [2] and in low dimensional strings [3]. In particular in the context of $c = 1$ strings, the cubic collective hamiltonian has been shown to reproduce perturbatively the correlation functions of the theory [4] and its exact fermionization [5] has been established in the potential free case [6].

We recall that the $c = 1$ string interpretation is obtained from the quantum mechanics of an hermitean matrix with (inverted harmonic oscillator) potential

$$v_b = \mu - \frac{1}{2}x^2 \quad (1)$$

in the double scaling limit [7,8] $N \rightarrow \infty$ and $\mu \rightarrow 0$ with $N\mu$ constant.

At the classical level, it is known that the collective field is the density variable description of the dynamics of the N eigenvalues of the matrix, which is described by a Calogero type [9] hamiltonian. This interplay between the collective field description of matrix quantum mechanics and Calogero models has played a fundamental role in the understanding of single matrix theories: for instance it has enabled the identification of their large N limit with the classical equations of motion of the theory subjected to specific initial conditions [10], and the fact that the Calogero model with harmonic type potentials is exactly integrable [9] is at the root of the uncovering of the W_∞ algebra in the $c = 1$ string [11,12].

A supersymmetric extension of the collective field theory has been proposed by Jevicki and one of the authors based on the metric structure of the cubic collective field theory [13]. It is perhaps not surprising in the light of the discussion above that at the classical level this extension corresponds to a density variable description of super-Calogero type models first discussed by Friedman and Mende [14] who specifically considered the harmonic potential

$$v'(x) = \omega x \quad (2)$$

In a seemingly unrelated development, Marinari and Parisi [15] introduced a supersymmetric extension of single matrix models by supertriangulation. It turns out, however, that the restriction of this model to a suitable invariant subspace [16] yields a system of the super-Calogero type. Therefore all three approaches are essentially equivalent, as it has been established in [17].

¹Recall that the relationship between the potential $v'(x)$ of the supersymmetric model and the bosonic potential v_b is $v_b = \frac{1}{2}v'^2(x) + \dots$

It was hoped that these approaches might provide a non-perturbative formulation of the $\hat{c} = 1$ superstring, but this has (so far) turned out not to be possible. As a matter of fact, it was established [17] that for a large class of potentials one cannot generate an infinite Liouville-like dimension as is the case of the $c = 1$ model. This is ultimately because it is hard to reconcile the positive definiteness of a supersymmetric theory with the (essentially) negative definite critical inverted harmonic oscillator potential of a $c = 1$ theory and the existence of a freely adjustable chemical potential parameter μ . From a string theory point of view, these supersymmetric extensions are perhaps more relevant to the stabilization of $d = 0$ models, in a well known mechanism of dimensional reduction [18] (for related developments based on loop equations, please refer to [19]; see also [31,32] for recent discussions of other possible continuum limits).

Recently, Jevicki and Yoneya [20] suggested that a matrix model with potential

$$v_b = -\frac{1}{2}x^2 + \frac{1}{2}\frac{M}{x^2} \quad (3)$$

may be relevant to the study of two dimensional string theory in a black-hole background, in the double scaling limit $N \rightarrow \infty$ and $M \rightarrow 0$ with $N^2 M$ constant. The existence of a quantum mechanical model where black-hole physics can be addressed in a possibly non-perturbative way cannot be underestimated. Moreover, the matrix model with the potential (3) has recently been shown to have an exact, computable S-matrix with particle production [21] and is known to possess (with both $\pm x^2$ harmonic potential) an infinite Lie Algebra [22]. Also, earlier studies of the physically acceptable quantum mechanical solutions of the Schrodinger equation for the $\frac{1}{x^2}$ potential on the real line have been carried out [23]. Therefore the study of the deformation of the harmonic oscillator potential by a singular $\frac{1}{x^2}$ contribution is of great interest.

In this article we study the supersymmetric collective field theory for the potential

$$v'(x) = \omega x - \frac{\eta}{x} \quad (4)$$

This corresponds to a supersymmetrization of the matrix model with a bosonic potential which is, apart from a constant, given by

$$v_b = \frac{1}{2}\omega^2 x^2 + \frac{1}{2}\frac{\eta^2}{x^2}. \quad (5)$$

The potential (4) generalizes the potential (2) originally considered by Friedman and Mende [14]. Via the the supersymmetric collective field theory description, we obtain a systematic semiclassical expansion of the corresponding super-Calogero model which may potentially be of relevance to the study of superstring theory in a black-hole background.

After a brief review of supersymmetric collective field theory [13] in Section 2, we carry out the analysis of the leading large- N supersymmetry preserving configuration in Section 3. We show that a symmetric density of eigenvalues exists which is the minimum of the leading effective bosonic potential and it simultaneously satisfies the usual analytic properties of the corresponding $d = 0$ solution to the problem. The only other classical configuration known to satisfy both requirements is that of the harmonic oscillator potential (2). This symmetric leading large N configuration turns out to have been previously discussed by Tan [24] in a different context. Using a classical result of Calogero [25], we show that the eigenvalues of the matrix correspond to the zeroes of an appropriate Laguerre polynomial, generalizing a corresponding result for the harmonic oscillator case and the zeroes of Hermite polynomials [10]. In Section 4, we discuss small fluctuations about the leading large N configuration and show that the spectrum is linear. We exhibit supersymmetry explicitly by identifying a majorana fermion as the partner of the scalar field. We also construct explicitly the polynomial eigenfunctions which are seen to be related to the Chebyshev polynomials, thereby generalizing a class of polynomial solutions discussed by Calogero [26]. In Section 5, we discuss interactions and reserve Section 6 for further discussion and conclusions. We believe that the results presented in this article provide the first collective field description of a two band solution beyond tree level (multiband models have been described at tree level and via orthogonal polynomials in ref [27]).

II. COLLECTIVE FIELD THEORY OF A SUPER SYMMETRIC MATRIX MODEL

In [13] the bosonic collective lagrangian:

$$L = \int dx \left(\frac{1}{2} \frac{(\partial_x^{-1} \dot{\phi})^2}{\phi} - \frac{\pi^2}{6} \phi^3(x, t) - v(x) \phi \right) \quad (6)$$

was supersymmetrized exploiting the metric structure of the kinetic energy term ($\phi \equiv \partial_x \varphi$)

$$\begin{aligned} L_T &= \frac{1}{2} \int dx \frac{(\dot{\varphi})^2}{\phi} \\ &= \int dx \int dy \dot{\varphi}(x, t) g_{xy}(\varphi) \dot{\varphi}(y, t) \end{aligned} \quad (7)$$

The resulting supersymmetric lagrangian is

$$\begin{aligned} L &= \frac{1}{2} \int \frac{dx}{\phi} \dot{\varphi}^2 - \frac{1}{2} \int dx \phi (W_{;x})^2 \\ &\quad + \frac{i}{2} \int \frac{dx}{\phi} (\psi^\dagger \dot{\psi} - \dot{\psi}^\dagger \psi) \\ &\quad + \frac{i}{2} \int \frac{dx}{\phi} \dot{\varphi} [\partial_x (\frac{\psi^\dagger}{\phi}) \phi - \psi^\dagger \partial_x (\frac{\psi}{\phi})] \\ &\quad + \int \frac{dx}{\phi} \psi^\dagger \partial_x W_{;x} \psi \\ &\quad - \int dx \int dy \psi^\dagger(x) W_{;xy} \psi(y) \end{aligned} \quad (8)$$

where the superpotential $W(x)$ is given by

$$W(x) = \int dx v(x) \phi - \frac{1}{2} \int dx \int dy \ln|x-y| \phi(x) \phi(y) \quad (9)$$

and $W_{;x} \equiv \partial_x \delta W[\phi] / \delta \phi(x) = \delta W / \delta \varphi(x)$. This lagrangian is accompanied by the constraint

$$\int dx \phi(x) = N \quad (10)$$

The lagrangian (8) is singular which is easily seen by computing the conjugate momenta of the fermionic fields. Due to the presence of the constraints

$$\chi = \Pi - \frac{i}{2} \frac{\psi^\dagger}{\phi}, \quad \bar{\chi} = \Pi^\dagger + \frac{i}{2} \frac{\psi^\dagger}{\phi} \quad (11)$$

quantization is achieved with the Dirac brackets

$$\{A, B\}^D = \{A, B\} - \sum_{i,j} \{A, \chi_i\} (\{\chi_i, \chi_j\}^{-1})_{ij} \{\chi_j, B\}. \quad (12)$$

Explicitly, after quantizing, one finds [13]:

$$[p(x), \varphi(y)] = -i\delta(x-y) \quad (13)$$

$$[p(x), \psi^\dagger(y)] = \frac{i}{2} \frac{\psi^\dagger}{\phi}(y) \partial_x \delta(x-y) \quad (14)$$

$$[p(x), \psi(y)] = \frac{i}{2} \frac{\psi}{\phi}(y) \partial_x \delta(x-y) \quad (15)$$

$$\{\psi(x), \psi^\dagger(y)\} = \phi(x) \delta(x-y) \quad (16)$$

The hamiltonian takes the simple form

$$\begin{aligned}
H = & \frac{1}{2} \int dx \phi(p(x) - \frac{i}{2} [\partial_x(\frac{\psi^\dagger}{\phi}) \frac{\psi}{\phi} - \frac{\psi^\dagger}{\phi} \partial_x(\frac{\psi}{\phi})])^2 \\
& + \frac{1}{2} \int dx \phi(W_{;x})^2 \\
& - \int \frac{dx}{\phi} \psi^\dagger \partial_x W_{;x} \psi + \int dx \int dy \psi^\dagger(x) W_{;xy} \psi(y)
\end{aligned} \tag{17}$$

and the corresponding supercharges are

$$\begin{aligned}
Q = & \int dx \psi^\dagger(x) [(p(x) - \\
& \frac{i}{2} [\partial_x(\frac{\psi^\dagger}{\phi}) \frac{\psi}{\phi} - \frac{\psi^\dagger}{\phi} \partial_x(\frac{\psi}{\phi})] - iW_{;x}] \\
Q^\dagger = & \int dx \psi(x) [(p(x) - \\
& \frac{i}{2} [\partial_x(\frac{\psi^\dagger}{\phi}) \frac{\psi}{\phi} - \frac{\psi^\dagger}{\phi} \partial_x(\frac{\psi}{\phi})] + iW_{;x}]
\end{aligned} \tag{18}$$

The above commutation relations (13) - (16) are unusual in that the bosonic momentum does not commute with the fermionic fields. This is not however, hard to understand. Classically, the above fields correspond to a density description of a super Calogero model [10,14]:

$$\phi(x) = \sum_i \delta(x - \lambda_i) \tag{19}$$

$$\phi\sigma(x) = - \sum_i \delta(x - \lambda_i) p_i \tag{20}$$

$$\psi(x) = - \sum_i \delta(x - x_i) \psi_i \tag{21}$$

$$\psi^\dagger(x) = - \sum_i \delta(x - x_i) \psi_i^\dagger. \tag{22}$$

where $\dot{\phi} = \phi\sigma$. The λ_i 's are N bosonic co-ordinates, p_i are the corresponding momenta and ψ_i and ψ_i^\dagger are fermionic (Grassman) superpartners. It is easy to see that (13) - (16) also follows from these density fields.

The commutators (13) - (16) are brought into a more familiar form by rescaling [17] $\psi(x) \rightarrow \sqrt{\phi(x)}\psi(x)$ and $\psi^\dagger(x) \rightarrow \sqrt{\phi(x)}\psi^\dagger(x)$. The new commutators are:

$$[\varphi(x), \varphi(y)] = 0 \tag{23}$$

$$[\varphi(x), p(y)] = i\delta(x - y) \tag{24}$$

$$\{\psi(x), \psi^\dagger(y)\} = \delta(x - y) \tag{25}$$

$$\begin{aligned}
[\varphi(x), \psi(y)] &= [\varphi(x), \psi^\dagger(y)] = [p(x), \psi(y)] = \\
&= [p(x), \psi^\dagger(y)] = 0
\end{aligned} \tag{26}$$

The hamiltonian becomes

$$\begin{aligned}
H = & \frac{1}{2} \int \frac{dx}{\phi} \left(\phi p - \frac{i}{2} [(\partial_x \psi^\dagger) \psi - \psi^\dagger (\partial_x \psi)] \right)^2 \\
& + \frac{1}{2} \int dx \phi(W_{;x})^2 - \frac{1}{2} \int dx [\psi^\dagger, \psi] \partial_x W_{;x} \\
& \frac{1}{2} \int dx \int dy [\psi^\dagger(x), \psi(y)] \sqrt{\phi(x)} W_{;xy} \sqrt{\phi(y)}
\end{aligned} \tag{27}$$

and the supercharges:

$$Q = \int dx \psi^\dagger(x) \sqrt{\phi(x)} \left(p(x) - \frac{i}{2\phi} \left((\partial_x \psi^\dagger) \psi - \psi^\dagger (\partial_x \psi) \right) + i v'(x) - i \oint dy \frac{\partial_y \varphi}{x-y} \right) \quad (28)$$

$$Q^\dagger = \int dx \psi(x) \sqrt{\phi(x)} \left(p(x) - \frac{i}{2\phi} \left((\partial_x \psi^\dagger) \psi - \psi^\dagger (\partial_x \psi) \right) - i v'(x) + i \oint dy \frac{\partial_y \varphi}{x-y} \right) \quad (29)$$

The square root factors are treated by expanding about the large N background configuration, which clearly generates an infinite perturbative expansion.

Now, we turn our attention to the N dependence of the Hamiltonian and the supercharges. It is well known that the Feynman diagrams of a matrix theory can be topologically classified according to their genus [28]. Writing the superpotential \bar{v} in terms of the supermatrix Φ [15,17]

$$\bar{v}(\Phi) = \text{Tr} \left(g_2 \Phi^2 + \frac{g_3}{\sqrt{N}} \Phi^3 + \dots + \frac{g_p}{N^{\frac{p}{2}-1}} \Phi^p \right) \quad (30)$$

it then follows that a diagram of genus Γ carries a factor $N^{2-2\Gamma}$. It is clear that for \bar{v} of the above form, the expressions $\frac{v(\sqrt{N}x)}{\sqrt{N}}$ and $v'(\sqrt{N}x)$ are both independent of N . Thus rescaling $x \rightarrow \sqrt{N}x, v \rightarrow \frac{v}{\sqrt{N}}$ and $\phi \rightarrow \sqrt{N}\phi$, our constraint on ϕ becomes N independent ($\int dx \phi(x) = 1$) and the N dependence of the Hamiltonian is made explicit:

$$\begin{aligned} H = & \frac{N^2}{2} \int dx \phi(x) \left(\oint dy \frac{\partial_y \varphi}{x-y} - v'(x) \right)^2 \\ & + \frac{1}{2} \int dx \left[\psi^\dagger(x) \sqrt{\phi(x)}, \frac{d}{dx} \oint dy \frac{\psi(y) \sqrt{\phi(y)}}{x-y} \right] \\ & - \frac{1}{2} \int dx [\psi^\dagger, \psi] \frac{d}{dx} \left(\oint dy \frac{\partial_y \varphi}{x-y} - v'(x) \right) \\ & + \frac{1}{2N^2} \int \frac{dx}{\phi} \left(\phi p - \frac{i}{2} \left[(\partial_x \psi^\dagger) \psi - \psi^\dagger (\partial_x \psi) \right] \right)^2 \end{aligned} \quad (31)$$

Requiring that supersymmetry is preserved to leading order, we obtain an integral equation for the vacuum configuration of the collective field:

$$\oint dy \frac{\phi_0(y)}{x-y} - v'(x) = 0 \quad (32)$$

Now, expand about the vacuum configuration ϕ_0 as:

$$\phi = \phi_0 + \frac{1}{N} \partial_x \eta \quad (33)$$

and rescale $p \rightarrow Np$. The above rescaling of the bosonic momentum, together with the $\frac{1}{N}$ coefficient of $\partial_x \eta$ ensures that all bosonic propagators are of order unity. Thus all explicit N dependence is absorbed into the vertices. This rescaling is purely a matter of convenience and does not change the theory.

III. THE VACUUM DENSITY

In this section we determine the (classical) vacuum for the superpotential (9) with

$$v(x) = \frac{1}{2} \omega x^2 - \eta N \log|x| \quad (34)$$

The explicit factor of N in the above equation is crucial for the identification of the large N expansion with an expansion in the genus of the diagrams of the original supermatrix model (see the comments following (30)). We rescale as explained in these comments, and are lead to the following integral equation for the vacuum configuration of the collective field

$$\oint dy \frac{\phi(y)}{x-y} = v(x) = \omega x - \frac{\eta}{x} \quad (35)$$

with $\int dx \phi = 1$. We are interested in the solution for which x runs from $-\infty$ to $+\infty$. This solution has been considered by Tan [24] in the context of the $d = 0$ generalized Penner models and we repeat the derivation here. $v(x)$ has two minima, situated symmetrically about the origin. These minima are separated by an infinitely high potential barrier such that the tunneling probability from one region to the other is zero. We therefore demand that ϕ has support only over the two finite intervals, $y_-^2 \leq y^2 \leq y_+^2$, around the minima of $v(x)$ on the real y axis. Thus, ϕ must not have a pole at the origin, since no eigenvalues can lie there. As is standard, we now continue into the complex plane, and denote the resulting function $G(z)$

$$G(z) = \oint dy \frac{\phi(y)}{z-y} \quad (36)$$

It follows that $G(z)$ only has two pairs of branch points at y_{\pm} and $-y_{\pm}$. If we connect these pairs of branch points along the real axis, we obtain a real analytic function bounded everywhere on the first sheet. Finally we obtain a normalisation condition by considering the $|z| \rightarrow \infty$ limit, which upon using the constraint $\int dx \phi(x) = 1$ becomes $G(z) \sim \frac{1}{z}$. It is easy to show that a symmetric ansatz of the form :

$$G(z) = \omega z - \frac{\eta}{z} + \frac{d}{z} \sqrt{(z^2 - y_-^2)(z^2 - y_+^2)} \quad (37)$$

leads to a consistent solution, with the normalization condition fixing $d = -\omega$ and

$$y_{\pm}^2 = \frac{1+\eta}{\omega} \pm \frac{1}{\omega} \sqrt{1+2\eta} \quad (38)$$

Now, observe that if we evaluate G at $z^{\pm} = x \pm i\epsilon$ we obtain:

$$G(z^{\pm}) = v(x) \pm i\pi\phi(x) \quad (39)$$

from which it follows that the collective field is given by:

$$\phi(x) = \frac{1}{|x|\pi} \sqrt{(1+2\eta)} \times \sqrt{\left(1 - \frac{\omega^2}{1+2\eta} \left(x^2 - \frac{1+\eta}{\omega}\right)^2\right)} \quad (40)$$

We emphasize that in our case the density field $\phi(x)$ describes a double band of eigenvalues. Notice that this solution is an even function of x . The importance of the symmetry of the solution can now be seen as follows: making use of the identity [6]²

$$\int dx \phi(x) \left(\oint dy \frac{\phi(y)}{x-y} \right)^2 = \frac{\pi^2}{3} \int dx \phi^3(x) \quad (41)$$

and using the symmetry of the solution, which implies that

$$\int dx \frac{\phi_0}{x} = 0, \quad (42)$$

we obtain

² The principal value prescription implicit in this identity is different from the one recently used in ref [31].

$$\begin{aligned}
V_{eff}[\phi] &= \frac{1}{2} \int dx \phi(x) \left[\oint dy \frac{\phi(y)}{x-y} - \left(\omega x - \frac{\eta}{x} \right) \right]^2 \\
&= \frac{\pi^2}{6} \int dx \phi_0^3(x) - \frac{\omega}{2} \left(\int dx \phi_0(x) \right)^2 \\
&\quad + \frac{1}{2} \int dx \left(\omega x - \frac{\eta}{x} \right)^2 \phi(x)
\end{aligned} \tag{43}$$

Thus the condition for the classical vacuum

$$\left(\frac{\delta V_{eff}}{\delta \phi} \right)_{\phi=\phi_0} = 0 \tag{44}$$

is equivalent to

$$\pi^2 \phi_0^2 = 2\omega + 2\omega\eta - \omega^2 x^2 - \frac{\eta^2}{x^2} \tag{45}$$

(the right hand side now corresponds to a "d = 1 potential") and reduces to a simple algebraic identity. This is a direct consequence of the symmetry of the classical vacuum. It is easy to verify that the solution to (44) is in agreement with the analytic solution (40).

We now give the collective field (40) an interpretation in terms of the density of zeroes of certain Laguerre polynomials. In terms of the original eigenvalue variables of the model, the equation for the vacuum (35) is:

$$\sum_{l=-\frac{N}{2}, l \neq m}^{\frac{N}{2}} \frac{1}{x_m - x_l} = \omega x_m - \frac{\eta}{x_m} \tag{46}$$

To solve this equation, we use the results of Calogero's work [23], based on the classical results of Stieltjes [30]. They show that the zeroes of the Laguerre polynomial $L_{\frac{N}{2}}^\alpha(x)$ obey:

$$\sum_{l=1, l \neq m}^{\frac{N}{2}} \frac{1}{x_m - x_l} = \frac{1}{2} \left(1 - \frac{1+\alpha}{x_m} \right) \tag{47}$$

Using the symmetry of the solution to write

$$\begin{aligned}
\sum_{n=-\frac{N}{2}, n \neq m}^{\frac{N}{2}} \frac{1}{x_m - x_n} &= \sum_{n=1, n \neq m}^{\frac{N}{2}} \left[\frac{1}{x_m - x_n} + \frac{1}{x_m + x_n} \right] \\
&= 2x_m \sum_{n=1, n \neq m}^{\frac{N}{2}} \frac{1}{x_m^2 - x_n^2}
\end{aligned} \tag{48}$$

we find that the zeroes of $L_{\frac{N}{2}}^{\eta-1}(\omega x^2)$ obey (46). Thus we see that the collective field ϕ_0 in fact describes the density of zeroes of the Laguerre polynomials in the limit that these polynomials have an infinite number of nodes. In the case of the harmonic oscillator ($\eta = 0$) Jevicki and Levine [10] proved explicitly that the collective field describes the density of zeroes of the Hermite polynomials which are of course special cases of the Laguerre polynomials.

IV. THE SPECTRUM

In this section we study the spectrum of the model. When supersymmetry is not broken in the leading order, we can expand as in (33) and obtain the following bosonic and fermionic quadratic contributions to the Hamiltonian

$$\begin{aligned}
H_0^B &= \frac{1}{2} \int dx \phi_0 p^2 - \frac{1}{2} \oint dx dy \frac{v(x) - v(y)}{x-y} \partial_x \eta \partial_y \eta \\
&\quad + \frac{\pi^2}{2} \int dx \phi_0 (\partial_x \eta)^2 \\
H_0^F &= \frac{1}{2} \int dx \left[\psi^\dagger(x) \sqrt{\phi_0(x)}, \frac{d}{dx} \oint dy \frac{\psi(y) \sqrt{\phi_0(y)}}{x-y} \right]
\end{aligned} \tag{49}$$

Considering the bosonic sector of the quadratic Hamiltonian, we see that apart from the term

$$-\frac{1}{2} \int dx dy \frac{v(x) - v(y)}{x - y} \partial_x \eta \partial_y \eta \quad (50)$$

we have just got the terms associated in the bosonic string theory to a massless scalar particle [3]. This term could, in principle, modify the spectrum in an unexpected way. For the potential we consider here, using the fact that $\int dx \partial_x \eta = 0$, this term may be simplified to:

$$-\frac{1}{2} \left[\int dx \frac{\partial_x \eta}{x} \right]^2 \quad (51)$$

This term can be made to vanish if we systematically restrict ourselves to odd fluctuations η , which we assume from now on. Notice that from (33) this simply implies that our density field $\phi(x)$ is always even, which is natural in view of the ground state of the previous section. Thus, under this restriction, the bosonic spectrum is that of a massless scalar. In order to exhibit this explicitly, one changes to the time of flight variable q [3]

$$dq = \frac{dx}{\phi_0} \quad q = \int \frac{dx}{\phi_0(x)} \quad (52)$$

and rescales

$$p \rightarrow \frac{p}{\phi_0} \quad \psi \rightarrow \frac{\psi}{\sqrt{\phi_0}} \quad (53)$$

The change of co-ordinates (52) is easily integrated to yield an explicit expression for q :

$$q = \frac{\pi}{2\omega} \arccos \left[\frac{-\omega}{\sqrt{1+2\eta}} \left(x^2 - \frac{1+\eta}{\omega} \right) \right] \quad (54)$$

where in the above, we have chosen $q = 0$ to correspond to $x^2 = y_-^2$.

A few remarks about the new parametrization (52) are in order. First, recall that in x space ϕ_0 is defined on two disconnected intervals. Since we have picked $q = 0$ to correspond to $x^2 = y_-^2$, ϕ_0 is defined on a single interval in q space. In terms of the new co-ordinate q , integration from y_- to y_+ corresponds to integrating from $q = 0$ to $q = L = \frac{\pi^2}{2\omega}$, and integration from $-y_+$ to $-y_-$ corresponds to integrating from $q = -L$ to $q = 0$. Explicitly

$$x(q) = \epsilon(q) \left[\frac{1+\eta}{\omega} - \frac{\sqrt{1+2\eta}}{\omega} \cos \left(\frac{2\omega q}{\pi} \right) \right]^{\frac{1}{2}}, \quad -L \leq q \leq L \quad (55)$$

In q -space, for odd fluctuations η , we obtain:

$$H_0^B = \frac{1}{2} \int dq \left(p^2 + \pi^2 (\partial_q \eta)^2 \right) \quad (56)$$

If we now expand our fields in terms of the oscillators

$$\eta(q) = \sum_{n=1}^{\infty} \frac{1}{\sqrt{4\pi^2 n}} (a_n + a_n^\dagger) \sin \frac{n\pi q}{L} \quad (57)$$

$$p(q) = \sum_{n=1}^{\infty} -i \sqrt{\frac{\pi^2 n}{4L^2}} (a_n - a_n^\dagger) \sin \frac{n\pi q}{L} \quad (58)$$

where $[a_m, a_n] = \delta_{mn}$, we obtain

$$H_0^B = \sum_{n=1}^{\infty} n\omega (a_n^\dagger a_n + \frac{1}{2}) \quad (59)$$

The fermionic sector of the quadratic Hamiltonian in q -space is

$$H_0^F = \frac{1}{2} \int_{-L}^L dq \left[\psi^\dagger(q), \frac{d}{dq} \oint_{-L}^L dq' \frac{\phi_0(q') \psi(q')}{x(q) - x(q')} \right] \quad (60)$$

The identity:

$$\oint_{-L}^L dq' \frac{\phi_0(q') \psi(q')}{x(q) - x(q')} = \oint_{-L}^L dq' \frac{\phi_0(q') x(q') \psi(q')}{x^2(q) - x^2(q')} \quad (61)$$

holds for our potential $v(x)$, provided we restrict ourselves to odd fermionic fluctuations ψ . On the left hand side of the above equation, there is a singularity of the form $\sqrt{\frac{1}{q}}$ at $q = -\frac{i\pi}{2\omega} \log \frac{1}{\sqrt{1+2\eta}}$ arising from the $\frac{1}{x}$ factor in $\phi_0(x)$. This singularity does not contribute for odd fermionic fluctuations ψ which is evident since, the factor $x(q')$ cancels this singularity on the right hand side of the above equation. Thus we again see the importance of the symmetry of the solution.

To determine the fermionic sector of the spectrum, introduce the oscillators

$$\begin{aligned} \psi(q) &= \frac{1}{\sqrt{2L}} \sum_{n=1}^{\infty} b_n \sin \frac{n\pi q}{L} \\ \psi^\dagger(q) &= \frac{1}{\sqrt{2L}} \sum_{n=1}^{\infty} b_n^\dagger \sin \frac{n\pi q}{L} \end{aligned}$$

where $\{b_m, b_n^\dagger\} = \delta_{mn}$. From (61), we find that we need to compute integrals of the form:

$$\oint_{-L}^L dq' \frac{\phi_0(q') x(q') \sin(\frac{n\pi q'}{L})}{x^2(q) - x^2(q')} \quad (62)$$

In the above replace $\sin(\frac{n\pi q'}{L})$ by $\frac{1}{i} e^{\frac{in\pi q'}{L}}$. This integral is now easily computed using contour integration arguments. Exploit the fact that the integrand is periodic with period $2L$, by closing the contour in the upper half plane, along the lines $q = -L$ and $q = L$. Close the box at infinity, parallel to the q axis. The only poles that contribute are on the real line at q and $-q$. This is only true for restricted fluctuations, since in general the $\frac{1}{x}$ singularity would contribute. We employ a principal value prescription which corresponds to taking one half of the residues of poles lying on the contour. The integrals are now easily computed, which together with the bosonic contribution gives the quadratic Hamiltonian:

$$H_0 = \sum_{n=1}^{\infty} n\omega (a_n^\dagger a_n + b_n^\dagger b_n) \quad (63)$$

thus explicitly demonstrating the supersymmetry of the semiclassical spectrum.

The above discussion has been carried out in q space, or time of flight variable. Following [13] the x space kernel that corresponds to the second of (49), can be written as

$$\int dx \oint dy \frac{[\psi^\dagger(x), \psi(y)]}{(x-y)^2} \phi_0(y) \quad (64)$$

Thus we see that by studying the fermionic fluctuations, we are naturally led to study the operator:

$$(O\psi)(x) = \oint dy \frac{\psi(y) \phi_0(y)}{(x-y)^2} \quad (65)$$

The above integral runs over the two disconnected bands of eigenvalues described by $\phi_0(x)$. The eigenfunctions are most easily determined by performing this integral in q space:

$$\begin{aligned} \oint dy \frac{\psi(y) \phi_0(y)}{(x-y)^2} &= \frac{d}{dx(q)} \oint_{-L}^L dq' \frac{\phi_0^2(q') \psi(q')}{x(q') - x(q)} \\ &= \frac{1}{\phi_0(q)} \frac{d}{dq} \oint_{-L}^L dq' \frac{\phi_0^2(q') \psi(q')}{x(q') - x(q)} \end{aligned} \quad (66)$$

Making the ansatz $\psi(q)\phi_0(q) = \sin \frac{n\pi q}{L}$, this integral reduces to (62). Explicitly performing the integral, we find

$$\psi_n(q) = \frac{\sin(\frac{n\pi q}{L})}{\phi_0(q)} \quad (67)$$

is an eigenfunction with eigenvalue $n\omega$. Rewriting this in x space, we find

$$\psi_n(x) = \frac{\pi x}{\sqrt{1+2\eta}} B_{n-1}\left(\frac{1+\eta}{\sqrt{1+2\eta}} - \frac{\omega x^2}{\sqrt{1+2\eta}}\right) \quad (68)$$

with B_n a Chebyshev polynomial. Notice that these eigenfunctions are indeed odd on the two bands. This is not suprising, since in the case of the harmonic oscillator ($\eta = 0$), the eigenfunctions are simply the Chebyshev polynomials [13].

A class of integral operators of the type (65) has been studied by Calogero [23]. To make a connection with his work, rescale $x^2 \rightarrow \frac{\sqrt{1+2\eta}}{\omega} x^2$. The kernel (65) becomes, after some algebra

$$(O\psi)(x) = \frac{d}{dx} \int_{a_-}^{a_+} dy^2 \frac{\sqrt{1 - (y^2 - \frac{1+\eta}{\sqrt{1+2\eta}})^2} \psi(y)}{\pi y(x^2 - y^2)} \quad (69)$$

with $a_{\pm} = (1+2\eta)(\frac{1+\eta}{\sqrt{1+2\eta}} \pm 1)$. The above analysis shows that the eigenfunctions ψ_n

$$\psi_n(x) = x B_{n-1}\left(\frac{1+\eta}{\sqrt{1+2\eta}} - x^2\right) \quad (70)$$

of the kernel have integer eigenvalues. Explicitly, the first three eigenfunctions are:

$$\begin{aligned} \psi_1(x) &= x \\ \psi_2(x) &= x\left(\frac{1+\eta}{\sqrt{1+2\eta}} - x^2\right) \\ \psi_3(x) &= x\left(1 + 2\left(\frac{1+\eta}{\sqrt{1+2\eta}} - x^2\right)^2\right) \end{aligned}$$

This generalizes well known results involving kernels of Chebyshev polynomials with weight $\sqrt{1-x^2}$ [33].

V. INTERACTIONS

We have seen that by suitably restricting the fluctuations, one is able to avoid a singularity at the origin in q space, at the level of the quadratic Hamiltonian. We were able to show that supersymmetry is unbroken, by computing the spectrum. This must indeed be the case, since in [17], the authors argued that if a supersymmetric classical configuration can be found, then the semiclassical spectrum is supersymmetric. Under this restriction on the fluctuations, it turns out that this singularity does not contribute in *any* higher order interactions [29]. In this section, we illustrate this for the cubic interaction.

The cubic contribution to the Hamiltonian is:

$$\begin{aligned} H_3 &= \frac{1}{2N} \int dx (\partial_x \eta) p^2 \\ &\quad + \frac{1}{2N} \int dx \partial_x \eta \left(\int dy \frac{\partial_y \eta}{x-y} \right)^2 \\ &\quad - \frac{i}{2N} \int dx [(\partial_x \psi^\dagger) \psi - \psi^\dagger (\partial_x \psi)] p \\ &\quad - \frac{1}{2N} \int dx [\psi^\dagger, \psi] \frac{d}{dx} \int dy \frac{\partial_y \eta}{x-y} \end{aligned} \quad (71)$$

In q space, this becomes, after some algebra:

$$\begin{aligned}
H_3 = & \frac{1}{2N} \int_{-L}^L dq (\partial_q \eta) p^2 \\
& + \frac{1}{2N} \int_{-L}^L dq \partial_q \eta \left(\oint_0^L dq' \frac{2x(q) \partial_{q'} \eta(q')}{x^2(q') - x^2(q)} \right)^2 \\
& - \frac{i}{2N} \int \frac{dq}{\phi_0^2(q)} [(\partial_q \psi^\dagger) \psi - \psi^\dagger (\partial_q \psi)] p \\
& + \frac{1}{2N} \int_{-L}^L dq [\psi^\dagger, \psi] \oint_{-L}^L dq' \frac{\partial_{q'} \eta}{(x(q) - x(q'))^2}
\end{aligned} \tag{72}$$

Notice that in the above there are no factors of $\phi_0(q)$. Thus the $\sqrt{\frac{1}{q}}$ singularity does not appear. The only new poles are introduced by the $\frac{1}{\phi_0^2}$ factor appearing in the last term. These new poles all lie on the real line, at $q = 0, \pm L$ and the integrals are easily computed. The principal valued integrals are performed by closing as for (62). Our principal value prescription corresponds to taking one quarter of the residues of the poles lying on the corners ($q = \pm L$) of the contour. The cubic interaction in the oscillator basis is now easily found:

$$\begin{aligned}
H_3 = & \frac{1}{4N\sqrt{\pi L}} \left(\frac{1+\eta}{1+2\eta} \right) \sum'_{n,p,y,l>0} \sqrt{k_l k_n k_p} (|k_l + k_n + k_p| \\
& - |k_l - k_n - k_p|) a_n^\dagger a_p^\dagger a_l + \\
& + \frac{1}{8N\sqrt{\pi L}} \left(\frac{1+\eta}{1+2\eta} \right) \sum'_{n,p,y,l>0} \sqrt{k_l} (k_p - k_n) \\
& (|k_l + k_n + k_p| - |k_l + k_n - k_p| + |k_l - k_n + k_p| \\
& - |k_l - k_n - k_p|) b_n^\dagger b_p a_l^\dagger + h.c.
\end{aligned} \tag{73}$$

where the prime on the sum indicates a sum over even integers only, and *h.c.* denotes the hermitean conjugate.

VI. DISCUSSION AND CONCLUSIONS

We have carried out a study of the supersymmetric collective field theory in the case of a harmonic potential deformed by a singular term. The emphasis has been on the density description of the underlying super Calogero model. The restriction on the density consistent with supersymmetry has been identified and forces one to consider systematically two band ansatz. Several classical results of Calogero both at the level of leading classical configuration and fluctuations have been obtained.

In the context of Penner models Tan [24] identified a double scaling limit in terms of an analytically continued crossover of the two bands. This analytic continuation cannot be performed in our model. Therefore the existence of a double scaling limit in Marinari-Parisi type models that does not simply amount to stabilization of $d = 0$ models remains an open question.

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